Math 249 Lecture 17 Notes

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1 Extending RSK to the $\mathbb{N}^n$ Case

Recall from last lecture that the RSK algorithm should give a bijection between multisets of pairs of positive integers and pairs of standard young tableau of the same shape. So far, we have shown a bijection

$$S_n \to \coprod_{|\lambda|=n} SYT(\lambda) \times SYT(\lambda)$$

given by $w \mapsto (P, Q)$, a pair of an insertion tableau, P, and a recording tableau, Q.

We can generalize this to $\mathbb{N}^n$ in place of S_n by labeling repeated copies of the same number, and treating later copies of the same number as "less than" previous copies of the number. In this case, the insertion tableau is semi-standard. So we get an algorithm mapping

$$\mathbb{N}^n \to \coprod_{|\lambda|=n} SSYT(\lambda) \times SYT(\lambda).$$

In fact, this tells us that

$$\sum_{\text{words } w} x^w = \sum_{|\lambda|=n} f_\lambda s_\lambda(x) = p_1(x)^n,$$

where f_λ is the number of SYT of shape λ; $f_\lambda = \langle p_1(x)^n, s_\lambda(y) \rangle$.

1.1 Creating pairs from words

Take a word

$$w = 3_1\, 1_1\, 1_2\, 5_1\, 3_2\, 2_1\, 3_3,$$

where successive copies of the same number are labeled in increasing order. We can get a permutation

$$\sigma = 4\,1\,2\,7\,5\,3\,6,$$

where we just write the order of the elements. But we can restore this information by recording

$$a = 1\,1\,2\,3\,3\,3\,5,$$

which notes the number of each element. We also have $\sigma^{-1} = 2\,3\,6\,1\,5\,7\,4$. Actually, σ^{-1} and a contain the information because σ^{-1} tells you where to put the numbers.

So we have mapped a word in $\mathbb{N}^n$ to a pair (σ, a). The pairs (σ, a) must satisfy the conditions that $a_i \le a_{i+1}$ for all i and if $i \in D(\sigma^{-1})$, then $a_i < a_{i+1}$, where $D = \{i : \sigma^{-1}(i+1) < \sigma^{-1}(i)\} \subseteq [n-1]$ is the "descent set." Here, $D(\sigma^{-1}) = \{3, 6\}$.

So $\mathbb{N}^n$ is in bijection with these pairs.

1.2 Creating pairs from tableau

Now we'll do labeling for semi-standard tableau. We want to map a SSYT(λ) to sets of compatible pairs (T, a), where T is a SYT and s is a list of letters. Take the tableau

$$S = \begin{array}{|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-2} 3_1 & 3_2 \\ \hline 1_1 & 1_2 & 2_1 & 3_3 \\ \hline\end{array}$$

and send it to

$$T = \begin{array}{|c|c|c|c|}\cline{1-1} 7 \\ \cline{1-2} 4 & 5 \\ \hline 1 & 2 & 3 & 6 \\ \hline\end{array} \qquad\qquad a = 1\,1\,2\,3\,3\,3\,5.$$

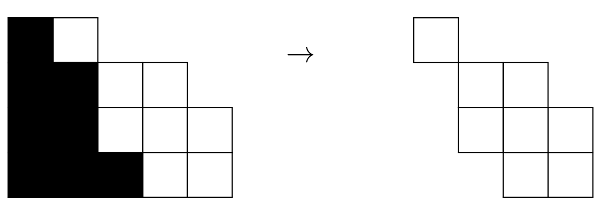
Define the descent set for tableau $D(T) = \{i : i+1 \text{ is above } i \text{ in } T\}$. In this case, $D(T) = \{3, 6\}$. The compatibility condition here is the same as before: that $a_i \le a_{i+1}$ and for $i \in D(\sigma^{-1})$, $a_i < a_{i+1}$.

To show that this is a bijection, we want to make sure this process is invertible. A run of consecutive entries in a SSYT is exactly a horizontal strip in a SYT.

1.3 Matching up the descent sets

We can match up the permutation with a tableau using our RSK algorithm from before, but do the pairs match up? We need to show that the descent sets are the same. In other words, we need to show that in RSK for S_n, $D(\sigma^{-1}) = D(P)$, where $\sigma \mapsto (P, Q)$.

Definition 1.1. A *skew diagram* is a Young diagram where we have removed a smaller partition.



What is a tableau on a completely disconnected skew diagra?m?

$$\begin{array}{ccccc} \boxed{3} & & & & \\ & \boxed{1} & & & \\ & & \boxed{5} & & \\ & & & \boxed{2} & \\ & & & & \boxed{4} \end{array}$$

If this is standard, then it is a permutation of $[n]$. Here, $\sigma = 3\,1\,5\,2\,4$. Then $D(\sigma^{-1}) = \{2, 4\}$, which is the same descent set of this tableau. So the descent set of a tableau is a generalization of the idea of the descent set of a permutation.

Say we're halfway through building the tableau from a permutation using RSK. We claim that the descent set never changes as we build it. For example, if $\sigma = 4\,1\,7\,3\,2\,8\,5\,6$, then we have

$$\begin{array}{ll} \boxed{4} & \\ \boxed{3} & \boxed{7} \\ \boxed{1} & \boxed{2} \end{array} \qquad \rightarrow \qquad \begin{array}{lll} \boxed{4} & & \\ \boxed{3} & \boxed{7} & \\ \boxed{1} & \boxed{2} & \boxed{8} \end{array}$$

$$\begin{array}{ccc} \boxed{8} & & \\ & \boxed{5} & \\ & & \boxed{6} \end{array} \qquad\qquad \begin{array}{cc} \boxed{5} & \\ & \boxed{6} \end{array}$$

If we add x to the tableau we're building, suppose it bumps y, where y is the least entry in the first row greater than x. If $y = x + 1$, then we bump y up a row; but this could only happen if y was already above x in our permutation skew tableau. If $y \geq x + 2$, then the descent set doesn't change when bumping y up a row because then $x + 1$ is not in the same row x was put in. And if x doesn't bump anything, the descent set doesn't change. We repeat this reasoning for each iteration of bumping numbers up the tableau. So the descent set really doesn't change.

We now claim that $D(Q) = D(\sigma)$, where $\sigma \mapsto (P, Q)$ in the SYT version of RSK. What this means is that if we insert x then y, then if $x < y$, then x is to the bottom right of y, and if $y > x$, then y is to the bottom right of x.

So we take a word, find the compatible pair, apply the first bijection on S_n, "remember" what the repeated entries were, and then compare the descent sets.

$$\begin{tikzcd} \mathbb{N}^n \arrow[d] & \coprod_{|\lambda|=n} SSYT(\lambda) \times SYT(\lambda) \\ S_n \arrow[r] & \coprod_{|\lambda|=n} SYT(\lambda) \times SSYT(\lambda) \arrow[u] \end{tikzcd}$$